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Bound states, Yangian symmetry and classical r-matrix for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring

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Abstract: We show that the recently found S-matrices describing the scattering of two-particle bound states of the light-cone string sigma model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ are compatible with Yangian symmetry. In case the invariance with respect to the centrally extended $\mathfrak{su}(2|2)$ algebra is not sufficient to fully specify the scattering matrix, the requirement of Yangian symmetry provides an alternative to the Yang-Baxter equation and leads to a complete, up to an overall phase, determination of the S-matrix. We then compare the semi-classical limit of the bound state S-matrices with the universal classical r-matrix by Beisert and Spill evaluated in the corresponding bound state representations and find perfect agreement.

Keywords: AdS-CFT Correspondence, Quantum Groups, Exact S-Matrix.

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1. Introduction

The discovery of integrable structures in the context of the AdS/CFT correspondence [1] has sparked many new insights and developments in this field. It was first noted that the operator spectrum of $\mathcal{N}=4$ super Yang-Mills theory can be linked to (integrable) spin chains [2]. The classical string sigma model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ was also shown to be integrable [3]. Although a complete proof of integrability of the spin chain associated to planar $\mathcal{N}=4$ super Yang-Mills theory and of its string dual is still missing, there is a lot of inspiring evidence that integrability is indeed preserved. Assuming integrability to hold has many important consequences. For example, the set of particle momenta is conserved and every scattering process factorizes into a sequence of two-body interactions. In other words, all the scattering information is encrypted in the two-body S-matrix.

As in many physical theories, symmetry algebras also play a crucial role here. The centrally extended $\mathfrak{su}(2|2)$ superalgebra has been shown to govern the asymptotic spectrum of the spin chain associated to planar $\mathcal{N}=4$ super Yang-Mills theory at higher loops [4, 5]. The very same algebra also emerges from string theory [6] as a symmetry algebra of the light-cone Hamiltonian [7, 8]. The requirement on the S-matrix to be invariant under the centrally extended $\mathfrak{su}(2|2)$ algebra fixes it uniquely up to an overall phase factor [5] and a choice of the representation basis [9]. With a proper choice of the scattering basis, the S-matrix exhibits most of the expected properties for a massive two-dimensional integrable field theory, including unitarity and crossing symmetry. It also obeys the Yang-Baxter equation [9].

The S-matrix approach [10, 11] was first developed in the spin chain framework of perturbative gauge theory. It allowed one to conjecture the corresponding "all-loop" Bethe equations describing the gauge theory asymptotic spectrum [5, 12]. On the string side, based on the knowledge of the classical finite-gap solutions [13], a Bethe ansatz for the $\mathfrak{su}(2)$ sector of the string sigma model was proposed [14]. The above mentioned non-analytic, overall (dressing) phase constitutes an important feature of the string S-matrix. It has been a subject of intensive research, see e.g. [15 – 18]. Most importantly, by combining its expansion in terms of local conserved charges with the requirement of crossing symmetry [19], one can find physically interesting solutions [15, 20], which nicely incorporate all available string and gauge theory data. The algebraic and the coordinate Bethe ansätze based on the string S-matrix have also been studied in [21, 22].

The study of the quantum/classical scattering matrices and their symmetries is also important for understanding finite size effects. Away from the infinite volume(charge) limit, wrapping interactions come into play and preclude the use of asymptotic Bethe equations. So far there are two attempts to deal with this problem. The first one consists in direct computation of finite-size corrections to the giant magnon [23] (or bound state) dispersion relation by using the sigma-model/algebraic curve approach [24 – 27]. Alternatively, these corrections can be obtained by using Lüscher's perturbative approach [28 – 31]. The second way [32] makes use of the thermodynamic Bethe ansatz [33]. One can define a mirror model [32] for which the finite size effects in the original theory are traded for finite temperature effects in the infinite volume. Both the Lüscher and the TBA approaches rely on the knowledge of the corresponding scattering matrices.

The two-body S-matrix that plays a pivotal role in the whole story actually has an even larger symmetry algebra[1] of Yangian type [34, 35]. Since Yangians have a number of useful properties, in particular, at the level of representation theory [36, 37], appearance of Yangian symmetry in the string context is quite a welcome feature. As a matter of fact, the existence of Yangian symmetry gives hope of constructing the universal R-matrix. Upon specifying suitable representations, this R-matrix would then reproduce various scattering processes; in particular, those involving the bound states.

At present, the existence of the universal R-matrix for the string sigma model is an open problem. On the other hand, there are two proposals for the classical r-matrix [38, 39], which might arise in the semi-classical limit of the yet to be found universal quantum R-matrix. The second proposal [39] was shown to arise from the canonical r-matrix of the exceptional algebra $\mathfrak{d}(2, 1; \epsilon)$ which is closely related to the $\mathfrak{su}(2|2)$ algebra [40]. From the string theory point of view, the classical r-matrix corresponds to the two-body S-matrix in the near plane-wave limit.

In addition to fundamental particles, the string sigma-model also contains bound states [41]. They fall into short (atypical) symmetric representations of the centrally extended $\mathfrak{su}(2|2)$ algebra [41 – 43]. In a recent work [44] the S-matrices $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$ which describe the scattering processes involving the fundamental multiplet (A) and the two-particle bound state multiplet (B) have been found. It appears that the extended $\mathfrak{su}(2|2)$

[1]See [9] for an earlier discussion of higher symmetries of the fundamental S-matrix.

symmetry together with the Yang-Baxter equations is sufficient to completely determine these S-matrices, up to an overall phase; the overall phase can be chosen to satisfy the additional requirement of crossing symmetry.

The aim of the present paper is to study the Yangian symmetry of the bound state S-matrices from [44] and also to compare them to the proposed classical r-matrix from [39] in the near plane-wave limit. We find that the S-matrices indeed respect Yangian symmetry. Moreover, as an alternative to the Yang-Baxter equation, Yangian symmetry completely determines the S-matrix $\mathbb{S}^{BB}$ up to a phase. Finally, upon comparing the proposed classical r-matrix to the bound state S-matrices in the near plane-wave limit, we find perfect agreement.

The paper is organized as follows. First, we recall the structure of the centrally extended $\mathfrak{su}(2|2)$ and the structure of its Yangian. Subsequently, we will discuss the formulation of the bound state representation in terms of differential operators that we used for our computations. In this language we specify coproducts of the $\mathfrak{su}(2|2)$ symmetry generators and of the Yangian generators and show that the bound state S-matrices respect Yangian symmetry. Last, we discuss the classical r matrix and compare it to the bound state S-matrices in the near plane-wave limit.

2. Centrally extended $\mathfrak{su}(2|2)$ and Yangians

The algebra which plays a key role in the entire discussion is centrally extended $\mathfrak{su}(2|2)$, which we will denote by $\mathfrak{h}$. It is the symmetry algebra of the light-cone Hamiltonian of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring and it also appears as the symmetry algebra of the spin chain connected to $\mathcal{N} = 4$ SYM. The algebra consists of bosonic generators $\mathbb{R}, \mathbb{L}$, supersymmetry generators $\mathbb{Q}, \mathbb{G}$ and central elements $\mathbb{H}, \mathbb{C}, \mathbb{C}^\dagger$. The non-trivial commutation relations between the generators are given by

$$
\begin{aligned}
[\mathbb{L}_a{}^b, \mathbb{J}_c] &= \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c & [\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma] &= \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma \\
[\mathbb{L}_a{}^b, \mathbb{J}^c] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c & [\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma] &= -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma \\
\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C} & \{\mathbb{G}_a{}^\alpha, \mathbb{G}_b{}^\beta\} &= \epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{C}^\dagger \\
\{\mathbb{Q}_\alpha^a, \mathbb{G}_b^\beta\} &= \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \frac{1}{2}\delta_b^a \delta_\alpha^\beta \mathbb{H}.
\end{aligned}
\tag{2.1}
$$

The first two lines show how the indices of an arbitrary generator with relevant indices transform. We will denote the eigenvalues of the central charges by $H, C, C^\dagger$. The charge H is Hermitian and the charges $C, C^\dagger$ are conjugate as well as the generators $\mathbb{Q}, \mathbb{G}$, i.e. $\mathbb{G} = \mathbb{Q}^\dagger$.

Let us now turn our attention to the (double) Yangian of centrally extended $\mathfrak{su}(2|2)$. We will briefly give the most relevant definitions and results, and refer to [36, 37] for more detailed accounts on Yangians in general and to [34, 39] for more details on the Yangian structure of $\mathfrak{h}$.

Double Yangian, generalities. The double Yangian $DY(\mathfrak{g})$ of a (simple) Lie algebra $\mathfrak{g}$ is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}[u, u^{-1}])$ of the loop algebra

$\mathfrak{g}[u, u^{-1}]$. Let us denote the deformation parameter by $\hbar$. The Yangian is generated by level n generators $\mathbb{J}^A_n$, $n \in \mathbb{Z}$ that satisfy the commutation relations

$$[\mathbb{J}^A_m, \mathbb{J}^B_n] = F^{AB}_C \mathbb{J}^C_{m+n} + \mathcal{O}(\hbar), \tag{2.2}$$

where F^{AB}_C are the structure constants of $\mathfrak{g}$. The level-0 generators $\mathbb{J}^A_0$ span the Lie-algebra. The coproduct is given by

$$\Delta(\mathbb{J}^A_n) = \mathbb{J}^A_n \otimes 1 + 1 \otimes \mathbb{J}^A_n + \frac{\hbar}{2}\sum_{m=0}^{n-1} F^A_{BC} \mathbb{J}^B_{n-1-m} \otimes \mathbb{J}^C_m. \tag{2.3}$$

Where the indices on the structure constants were lowered with the Cartan-Killing matrix.

The Yangian can be supplied with the structure of a quasi-cocommutative Hopf-algebra if there is an R-matrix, $R \in DY(\mathfrak{g}) \otimes DY(\mathfrak{g})$ such that

$$\Delta^{op}(\mathbb{J}^A_n) R = R \Delta(\mathbb{J}^A_n), \tag{2.4}$$

with Δ^{op} the opposite coproduct, $\Delta^{op} = P\Delta$, where P is the (graded) permutation operator. For conventional Yangians this universal R-matrix exists and can be explicitly constructed with the help of the Cartan-Killing matrix.

An important representation of the Yangian is the evaluation representation. This representation consists of states $|u\rangle$, with action $\mathbb{J}^A_n |u\rangle = u^n \mathbb{J}^A_0 |u\rangle$. Hence, upon choosing a representation of the Lie algebra we obtain a representation of the Yangian. The coproduct becomes particulary easy in this representation. Let it act on the state $|u_1\rangle \otimes |u_2\rangle$, then it is of the form:

$$\Delta(\mathbb{J}^A_n) \approx \frac{u_1^{n-1} - u_2^{n-1}}{u_1^{-1} - u_2^{-1}} \Delta(\mathbb{J}^A_0) + \frac{u_1^n - u_2^n}{u_1 - u_2} \Delta(\mathbb{J}^A_1). \tag{2.5}$$

This means that if one wants to check invariance of an R-matrix under Yangian symmetry in the evaluation representation it is enough to check this for $\mathbb{J}^A_0, \mathbb{J}^A_1$.

The parameter $\hbar$ is viewed as a quantum parameter and the Yangian as a quantum deformation of the enveloping algebra. We can consider the semi-classical limit by working consistently up to order $\hbar$. In this limit, the universal R-matrix expands as:

$$R = 1 + \hbar r + \mathcal{O}(\hbar^2). \tag{2.6}$$

The operator r is called the classical r-matrix.

Double Yangian, centrally extended $\mathfrak{su}(2|2)$. Unfortunately $\mathfrak{h}$ is not simple and hence the above discussed methods cannot be straightforwardly applied.

For the coproduct one needs to introduce a non-trivial braiding [34, 45, 46],

$$\begin{aligned} \Delta(\mathbb{J}^A_n) &= \mathbb{J}^A_n \otimes 1 + \mathcal{U}^{[A]} \otimes \mathbb{J}^A_n + \frac{\hbar}{2}\sum_{m=0}^{n-1} F^A_{BC} \mathbb{J}^B_{n-1-m} \mathcal{U}^{[C]} \otimes \mathbb{J}^C_m + \mathcal{O}(\hbar^2) \\ \Delta(\mathcal{U}) &= \mathcal{U} \otimes \mathcal{U}, \end{aligned} \tag{2.7}$$

for some Abelian generator $\mathcal{U}$ and "braid charges":

$$[\mathbb{C}^\dagger] = -2, \quad [\mathbb{G}] = -1, \quad [\mathbb{L}] = [\mathbb{R}] = [\mathbb{H}] = 0, \quad [\mathbb{Q}] = 1, \quad [\mathbb{C}] = 2. \tag{2.8}$$

In section 3.2 we will use another formulation of the coproduct which avoids the explicit use of the braiding factors. In order to make supply the Yangian with a quasi-cocommutative structure, it was shown that the central charges $\mathbb{C}, \mathbb{C}^\dagger$ need to be identified with the braiding factor $\mathcal{U}$ and $\mathbb{H}$ in the following way [34, 46]:

$$\begin{aligned} \mathbb{C}_0 &\sim g(1-\mathcal{U}^2) & \mathbb{C}_0^\dagger &\sim g(1-\mathcal{U}^{-2}) \\ \mathbb{C}_1 &\sim \mathbb{H}(1+\mathcal{U}^2) & \mathbb{C}_1^\dagger &\sim -\mathbb{H}(1+\mathcal{U}^{-2}). \end{aligned} \tag{2.9}$$

In the evaluation representation we can use this to express the evaluation parameter u in terms of the eigenvalues of H and the braiding operator:

$$\mathbb{J}_n^A = (iu)^n \mathbb{J}_0^A, \qquad iu \sim \mathbb{H}\frac{1+\mathcal{U}^2}{1-\mathcal{U}^2} \tag{2.10}$$

This is quite different from the standard case, where the evaluation parameter is unrelated to the algebra. The adjusted notion of coproduct raises the question whether a universal R-matrix can still be found in order to make the Yangian quasi-cocommutative. One could hope that, just as in the simple case, it could be constructed by means of the Cartan-Killing matrix. However, for $\mathfrak{h}$ this appears to be singular and hence the standard construction breaks down. Nevertheless, for the fundamental evaluation represenation of $\mathfrak{h}$, the R-matrix has been found as a scattering matrix [5, 9] and it indeed respects Yangian symmetry [34].

Although an expression for the universal R-matrix is currently lacking, there have been proposals for the classical r-matrix [38, 39]. We will focus on the proposal [39] in section 4.

3. Representation with differential operators and symmetry invariance

The representations that describe M-particle bound states are $4M$-dimensional and because of this, the sizes of the involved matrices quickly get out of hand. To avoid doing computations with unwieldy matrices it is useful to put this representation in the formalism of differential operators. The encountered totally symmetric representation of M-particle bound states can be identified with a $4M$-dimensional graded vector space of monomials of degree M and the different generators can be represented by corresponding differential operators [44].

3.1 Formalism and bound state representations

Consider the vector space of analytic functions of two bosonic variables w_a and two fermionic variables θ_α. Since we are dealing with analytic functions we can expand any such function $\Phi(w,\theta)$:

$$\begin{aligned} \Phi(w,\theta) &= \sum_{M=0}^{\infty} \Phi_M(w,\theta), \\ \Phi_M &= \phi^{a_1\ldots a_M} w_{a_1}\ldots w_{a_M} + \phi^{a_1\ldots a_{M-1}\alpha} w_{a_1}\ldots w_{a_{M-1}}\theta_\alpha \\ &\quad + \phi^{a_1\ldots a_{M-2}\alpha\beta} w_{a_1}\ldots w_{a_{M-2}}\theta_\alpha\theta_\beta. \end{aligned} \tag{3.1}$$

The representation of centrally extended $\mathfrak{su}(2|2)$, that describes M-particle bound states of the light-cone string theory has dimension $4M$. It is realized on a graded vector space with basis $|e_{a_1\ldots a_M}\rangle, |e_{a_1\ldots a_{M-1}\alpha}\rangle, |e_{a_1\ldots a_{M-2}\alpha\beta}\rangle$, where a_i are bosonic indices and α, β are fermionic indices and each of the basis vectors is totally symmetric in the bosonic indices and anti-symmetric in the fermionic indices [41, 42, 44]. In terms of the above analytic functions, the basis vectors of the totally symmetric representation can evidently be identified $|e_{a_1\ldots a_M}\rangle \leftrightarrow w_{a_1}\ldots w_{a_M}, |e_{a_1\ldots a_{M-1}\alpha}\rangle \leftrightarrow w_{a_1}\ldots w_{a_{M-1}}\theta_\alpha$ and $|e_{a_1\ldots a_{M-1}\alpha\beta}\rangle \leftrightarrow w_{a_1}\ldots w_{a_{M-2}}\theta_\alpha\theta_\beta$ respectively. In other words, we find the atypical totally symmetric representation describing M-particle bound states when we restrict to terms Φ_M.

In this representation the generators of $\mathfrak{h}$ can be written in differential operator form in the following way

$$\begin{aligned} \mathbb{L}_a{}^b &= w_a\frac{\partial}{\partial w_b} - \frac{1}{2}\delta_a^b w_c\frac{\partial}{\partial w_c}, & \mathbb{R}_\alpha{}^\beta &= \theta_\alpha\frac{\partial}{\partial\theta_\beta} - \frac{1}{2}\delta_\alpha^\beta\theta_\gamma\frac{\partial}{\partial\theta_\gamma}, \\ \mathbb{Q}_\alpha{}^a &= a\theta_\alpha\frac{\partial}{\partial w_a} + b\epsilon^{ab}\epsilon_{\alpha\beta}w_b\frac{\partial}{\partial\theta_\beta}, & \mathbb{G}_a{}^\alpha &= dw_a\frac{\partial}{\partial\theta_\alpha} + c\epsilon_{ab}\epsilon^{\alpha\beta}\theta_\beta\frac{\partial}{\partial w_b} \end{aligned} \tag{3.2}$$

and the central charges are

$$\begin{aligned} \mathbb{C} &= ab\left(w_a\frac{\partial}{\partial w_a} + \theta_\alpha\frac{\partial}{\partial\theta_\alpha}\right) & \mathbb{C}^\dagger &= cd\left(w_a\frac{\partial}{\partial w_a} + \theta_\alpha\frac{\partial}{\partial\theta_\alpha}\right) \\ \mathbb{H} &= (ad+bc)\left(w_a\frac{\partial}{\partial w_a} + \theta_\alpha\frac{\partial}{\partial\theta_\alpha}\right). \end{aligned} \tag{3.3}$$

To form a representations, the parameters a, b, c, d satisfy the condition $ad - bc = 1$. The central charges become M dependent:

$$H = M(ad+bc), \qquad C = Mab, \qquad C^\dagger = Mcd. \tag{3.4}$$

In what follows we will also need an additional operator[2]

$$\Sigma = \frac{1}{2}\frac{1}{ad+bc}\left(w_a\frac{\partial}{\partial w_a} - \theta_a\frac{\partial}{\partial\theta_\alpha}\right). \tag{3.5}$$

This operator corresponds (up to the prefactor) to the grading matrix Σ of [9] and it distinguishes the superfield components with different numbers of fermions. On bound state representations Σ has the following commutation relations with the algebra generators

$$\begin{aligned} [\Sigma, \mathbb{Q}^a{}_\beta] &= -\mathbb{Q}^a{}_\beta + 2\mathbb{C}\mathbb{H}^{-1}\epsilon_{\beta\gamma}\epsilon^{ad}\mathbb{G}^\gamma{}_d \\ [\Sigma, \mathbb{G}^\alpha{}_b] &= \mathbb{G}^\alpha{}_b - 2\mathbb{C}^\dagger\mathbb{H}^{-1}\epsilon^{\alpha\gamma}\epsilon_{bd}\mathbb{Q}^d{}_\gamma \\ [\Sigma, \mathbb{L}^a{}_b] &= [\Sigma, \mathbb{R}^\alpha{}_\beta] = [\Sigma, \mathbb{H}] = 0. \end{aligned} \tag{3.6}$$

We will also introduce the following quadratic operator

$$\mathcal{T} = \mathbb{R}_\beta{}^\alpha\mathbb{R}_\alpha{}^\beta - \mathbb{L}_b{}^a\mathbb{L}_a{}^b + \mathbb{G}_a^\alpha\mathbb{Q}_\alpha{}^a - \mathbb{Q}_\alpha{}^a\mathbb{G}_a{}^\alpha. \tag{3.7}$$

[2]We are grateful to G. Arutyunov for suggesting this.

The operators Σ and $\mathcal{T}$ can be used to construct the Casimir operator $\mathfrak{C}$ of the $\mathfrak{u}(2|2)$ algebra

$$\mathfrak{C} = \Sigma\mathbb{H} - \mathcal{T} \tag{3.8}$$

which in the M-particle bound state representation has the following eigenvalue

$$\mathfrak{C}|M\rangle = M(M-1)|M\rangle\,. \tag{3.9}$$

Further, we introduce the parameterization for a, b, c, d in terms of the particle momentum and the coupling g:

$$\begin{aligned} a &= \sqrt{\frac{g}{2M}}\eta & b &= \sqrt{\frac{g}{2M}}\frac{i\zeta}{\eta}\left(\frac{x^+}{x^-}-1\right) \\ c &= -\sqrt{\frac{g}{2M}}\frac{\eta}{\zeta x^+} & d &= \sqrt{\frac{g}{2M}}\frac{x^+}{i\eta}\left(1-\frac{x^-}{x^+}\right), \end{aligned} \tag{3.10}$$

where the parameters $x^\pm$ satisfy

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2Mi}{g}, \qquad \frac{x^+}{x^-} = e^{ip}. \tag{3.11}$$

Finally, the eigenvalue of the braid operator is found to be $\mathcal{U} = \sqrt{\frac{x^+}{x^-}}$ and the parameter of the evaluation representation is identified with $u_j = x_j^+ + \frac{1}{x_j^+} - \frac{iM_j}{g}$. The fundamental representation corresponds to taking $M = 1$.

The totally symmetric representation is now completely fixed by specifying $x^\pm, g, \eta, M$. The factor of η reflects a freedom in choosing basis vectors. However, as found in [32], it appears that string theory selects a particular choice of η, ζ:

$$\eta = e^{i\xi}e^{\frac{i}{4}p}\sqrt{ix^- - ix^+}, \qquad \zeta = e^{2i\xi}. \tag{3.12}$$

As a consequence of this choice, the S-matrix satisfies the normal, untwisted Yang-Baxter equation and is, in fact, a symmetric operator. Adopting this choice also has some consequences for the braiding factor in the coproduct. This will be discussed in the next sections.

3.2 Tensor products, coproducts and symmetries

When analyzing R-matrices or S-matrices one needs to consider (graded) tensor products of representations. In the context of differential operators in a superspace, this is easily realized by considering the product of two irreducible superfields $\Phi_{M_1}(w_a, \theta_\alpha)\Phi_{M_2}(u_a, \vartheta_\alpha)$ depending on different sets of coordinates.

Coproduct of the algebra generators. Let us now consider the coproducts of the symmetry generators of centrally extended $\mathfrak{su}(2|2)$. As discussed in [9], the S-matrix is a map between the following representations:

$$\mathbb{S}: \ \mathscr{V}_{M_1}(p_1, e^{ip_2}) \otimes \mathscr{V}_{M_2}(p_2, 1) \longrightarrow \mathscr{V}_{M_1}(p_1, 1) \otimes \mathscr{V}_{M_2}(p_2, e^{ip_1}), \tag{3.13}$$

where $\mathscr{V}_{M_i}(p_i, e^{2i\xi})$ is the M_i-bound state representation with parameters a_i, b_i, c_i, d_i with the explicit choice of $\zeta = e^{2i\xi}$. Taking into account the above parameters of the different representations, we see that when checking the relation

$$\mathbb{S}\ \Delta(\mathbb{J}_0^A) = \Delta^{op}(\mathbb{J}_0^A)\ \mathbb{S}, \tag{3.14}$$

all the explicit braiding factors drop out and we get

$$\Delta(\mathbb{J}_0^A) = \mathbb{J}_{1;0}^A + \mathbb{J}_{2;0}^A. \tag{3.15}$$

Here $\Delta^{op}(\mathbb{J}_i^A)$ acts on $\mathscr{V}_{M_1}(p_1,1)\otimes\mathscr{V}_{M_2}(p_2,e^{ip_1})$ and $\Delta(\mathbb{J}_i^A)$ acts on $\mathscr{V}_{M_1}(p_1,e^{ip_2})\otimes\mathscr{V}_{M_2}(p_2,1)$, with the appropriate coefficients a,b,c,d. We will give explicit expressions in the next section for each of these coefficients. In the above formula $\mathbb{J}_{i;0}^A$ is the operator $\mathbb{J}_0^A$ acting in the i-th space.

In [5, 9] the requirement of invariance under $\mathfrak{h}$, in the sense of commuting with (3.15), was found to be sufficient to fix the fundamental S-matrix, denoted by $\mathbb{S}^{AA}$, up to a phase factor. This procedure has also been carried out for the two-particle bound states representations [44]. This, together with Yang-Baxter, was again enough to fix the involved S-matrices up to a phase factor. For the explicit form of these S-matrices, denoted by $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$, we refer to [44].

We will spell out the phase factors, since they will play a role when comparing the S-matrices to the classical r-matrix. For $\mathbb{S}^{AA}$ the corresponding phase factor has been studied quite intensively. This factor allows one to derive the phase factors for $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$ by applying the fusion procedure [41, 48, 49].[3] Define the function

$$G(n) := \frac{u_1 - u_2 + \frac{in}{g}}{u_1 - u_2 - \frac{in}{g}}, \tag{3.16}$$

where $u_j = x_j^+ + \frac{1}{x_j^+} - \frac{iM_j}{g}$. The phase factors of the different matrices that follow from fusion and crossing symmetry are:

$$\begin{aligned} S_0^{AA} &= \sqrt{G(0)G(2)}\ \sqrt{\frac{x_{1;1}^- x_{2;1}^+}{x_{1;1}^+ x_{2;1}^-}}\ \sigma(x_{1;1}, x_{1;1}) \\ S_0^{AB} &= \sqrt{G(1)G(3)}\ \frac{x_{1;2}^-}{x_{1;2}^+}\sqrt{\frac{x_{2;2}^+}{x_{2;2}^-}}\ \sigma(x_{1;1}, x_{2;2}) \\ S_0^{BB} &= G(2)\sqrt{G(4)}\ \frac{x_{1;2}^- x_{2;2}^+}{x_{1;2}^+ x_{2;2}^-}\ \sigma(x_{1;2}, x_{2;2}), \end{aligned} \tag{3.17}$$

where $\sigma(p,q) = e^{i\theta(p,q)}$ is the dressing phase. The canonical S-matrices, completed with the above phases respect crossing symmetry [44].

[3] The fusion procedure for rational S/R-matrices based on $\mathfrak{gl}(m|n)$ has recently been worked out [47].

Yangian symmetry of the S-matrices. It appears that the coproduct of the Yangian generators can be written in the standard form. Let us denote the first Yangian generator of an operator $\mathbb{J}$ by $\hat{\mathbb{J}}$. All the braiding factors entering in the coproduct (2.7) never explicitly appear in the operator formalism discussed above. The coproduct in the differential operator form is now given by

$$\begin{aligned}
\Delta(\hat{\mathbb{L}}^a{}_b) &= \hat{\mathbb{L}}_{1;b}^{\,a} + \hat{\mathbb{L}}_{2;b}^{\,a} + \frac{1}{2}\mathbb{L}_{1;b}^{\,c}\mathbb{L}_{2;c}^{\,a} - \frac{1}{2}\mathbb{L}_{1;c}^{\,a}\mathbb{L}_{2;b}^{\,c} - \frac{1}{2}\mathbb{G}_{1;b}^{\,\gamma}\mathbb{Q}_{2;\gamma}^{\,a} - \frac{1}{2}\mathbb{Q}_{1;\gamma}^{\,a}\mathbb{G}_{2;b}^{\,\gamma} \\
&\quad + \frac{1}{4}\delta_b^a\mathbb{G}_{1;c}^{\,\gamma}\mathbb{Q}_{2;\gamma}^{\,c} + \frac{1}{4}\delta_b^a\mathbb{Q}_{1;\gamma}^{\,c}\mathbb{G}_{2;c}^{\,\gamma} \\
\Delta(\hat{\mathbb{R}}^\alpha{}_\beta) &= \hat{\mathbb{R}}_{1;\beta}^{\,\alpha} + \hat{\mathbb{R}}_{2;\beta}^{\,\alpha} - \frac{1}{2}\mathbb{R}_{1;\beta}^{\,\gamma}\mathbb{R}_{2;\gamma}^{\,\alpha} + \frac{1}{2}\mathbb{R}_{1;\gamma}^{\,\alpha}\mathbb{R}_{2;\beta}^{\,\gamma} + \frac{1}{2}\mathbb{G}_{1;c}^{\,\alpha}\mathbb{Q}_{2;\beta}^{\,c} + \frac{1}{2}\mathbb{Q}_{1;\beta}^{\,c}\mathbb{G}_{2;c}^{\,\alpha} \\
&\quad - \frac{1}{4}\delta_\beta^\alpha\mathbb{G}_{1;c}^{\,\gamma}\mathbb{Q}_{2;\gamma}^{\,c} - \frac{1}{4}\delta_\beta^\alpha\mathbb{Q}_{1;\gamma}^{\,c}\mathbb{G}_{2;c}^{\,\gamma} \qquad (3.18) \\
\Delta(\hat{\mathbb{Q}}^a{}_\beta) &= \hat{\mathbb{Q}}_{1;\beta}^{\,a} + \hat{\mathbb{Q}}_{2;\beta}^{\,a} - \frac{1}{2}\mathbb{R}_{1;\beta}^{\,\gamma}\mathbb{Q}_{2;\gamma}^{\,a} + \frac{1}{2}\mathbb{Q}_{1;\gamma}^{\,a}\mathbb{R}_{2;\beta}^{\,\gamma} - \frac{1}{2}\mathbb{L}_{1;c}^{\,a}\mathbb{Q}_{2;\beta}^{\,c} + \frac{1}{2}\mathbb{Q}_{1;\beta}^{\,c}\mathbb{L}_{2;c}^{\,a} \\
&\quad - \frac{1}{4}\mathbb{H}_1\mathbb{Q}_{2;\beta}^{\,a} + \frac{1}{4}\mathbb{Q}_{1;\beta}^{\,a}\mathbb{H}_2 + \frac{1}{2}\epsilon_{\beta\gamma}\epsilon^{ad}\mathbb{C}_1\mathbb{G}_{2;d}^{\,\gamma} - \frac{1}{2}\epsilon_{\beta\gamma}\epsilon^{ad}\mathbb{G}_{1;d}^{\,\gamma}\mathbb{C}_2 \\
\Delta(\hat{\mathbb{G}}^\alpha{}_b) &= \hat{\mathbb{G}}_{1;b}^{\,\alpha} + \hat{\mathbb{G}}_{2;b}^{\,\alpha} + \frac{1}{2}\mathbb{L}_{1;b}^{\,c}\mathbb{G}_{2;c}^{\,\alpha} - \frac{1}{2}\mathbb{G}_{1;c}^{\,\alpha}\mathbb{L}_{2;b}^{\,c} + \frac{1}{2}\mathbb{R}_{1;\gamma}^{\,\alpha}\mathbb{G}_{2;b}^{\,\gamma} - \frac{1}{2}\mathbb{G}_{1;b}^{\,\gamma}\mathbb{R}_{2;\gamma}^{\,\alpha} \\
&\quad + \frac{1}{4}\mathbb{H}_1\mathbb{G}_{2;b}^{\,\alpha} - \frac{1}{4}\mathbb{G}_{1;b}^{\,\alpha}\mathbb{H}_2 - \frac{1}{2}\epsilon_{bc}\epsilon^{\alpha\gamma}\mathbb{C}_1^\dagger\mathbb{Q}_{2;\gamma}^{\,c} + \frac{1}{2}\epsilon_{bc}\epsilon^{\alpha\gamma}\mathbb{Q}_{1;\gamma}^{\,c}\mathbb{C}_2^\dagger
\end{aligned}$$

and for the central charges:

$$\begin{aligned}
\Delta(\hat{\mathbb{H}}) &= \hat{\mathbb{H}}_1 + \hat{\mathbb{H}}_2 + \frac{1}{2}\mathbb{C}_1\mathbb{C}_2^\dagger - \frac{1}{2}\mathbb{C}_1^\dagger\mathbb{C}_2 \\
\Delta(\hat{\mathbb{C}}) &= \hat{\mathbb{C}}_1 + \hat{\mathbb{C}}_2 + \frac{1}{2}\mathbb{H}_1\mathbb{C}_2 - \frac{1}{2}\mathbb{C}_1\mathbb{H}_2 \qquad (3.19) \\
\Delta(\hat{\mathbb{C}}^\dagger) &= \hat{\mathbb{C}}_1^\dagger + \hat{\mathbb{C}}_2^\dagger + \frac{1}{2}\mathbb{H}_1\mathbb{C}_2^\dagger - \frac{1}{2}\mathbb{C}_1^\dagger\mathbb{H}_2.
\end{aligned}$$

The product is ordered, e.g. $\mathbb{Q}_1\mathbb{Q}_2$ means first apply $\mathbb{Q}_2$, then $\mathbb{Q}_1$. Also, in the evaluation representation we identify $\hat{\mathbb{J}} = \frac{g}{2i}u\mathbb{J}$. As stressed in the previous section, Δ^{op} acts on representations with different parameters ζ. For completeness we will explicitly give the parameters a, b, c, d for the involved representations. The coefficients for $\Delta(\mathbb{J})$ are given by:

$$\begin{aligned}
a_1 &= \sqrt{\frac{g}{2M_1}}\eta_1 & b_1 &= -ie^{ip_2}\sqrt{\frac{g}{2M_1}}\frac{1}{\eta_1}\left(\frac{x_1^+}{x_1^-}-1\right) \\
c_1 &= -e^{-ip_2}\sqrt{\frac{g}{2M_1}}\frac{\eta_1}{x_1^+} & d_1 &= i\sqrt{\frac{g}{2M_1}}\frac{x_1^+}{\eta_1}\left(\frac{x_1^-}{x_1^+}-1\right) \\
\eta_1 &= e^{i\frac{p_1}{4}}e^{i\frac{p_2}{2}}\sqrt{ix_1^- - ix_1^+} & & \qquad (3.20) \\
a_2 &= \sqrt{\frac{g}{2M_2}}\eta_2 & b_2 &= -i\sqrt{\frac{g}{2M_2}}\frac{1}{\eta_2}\left(\frac{x_2^+}{x_2^-}-1\right) \\
c_2 &= -\sqrt{\frac{g}{2M_2}}\frac{\eta_2}{x_2^+} & d_2 &= i\sqrt{\frac{g}{2M_2}}\frac{x_2^+}{i\eta_2}\left(\frac{x_2^-}{x_2^+}-1\right) \\
\eta_2 &= e^{i\frac{p_2}{4}}\sqrt{ix_2^- - ix_2^+}
\end{aligned}$$

The coefficients in $\Delta^{op}(\mathbb{J})$ are given by:

$$\begin{aligned}
a_1^{op} &= \sqrt{\frac{g}{2M_1}}\eta_1^{op} & b_1^{op} &= -i\sqrt{\frac{g}{2M_1}}\frac{1}{\eta_1^{op}}\left(\frac{x_1^+}{x_1^-}-1\right)\\
c_1^{op} &= -\sqrt{\frac{g}{2M_1}}\frac{\eta_1^{op}}{x_1^+} & d_1^{op} &= i\sqrt{\frac{g}{2M_1}}\frac{x_1^+}{i\eta_1^{op}}\left(\frac{x_1^-}{x_1^+}-1\right)\\
\eta_1^{op} &= e^{i\frac{p_1}{4}}\sqrt{ix_1^- - ix_1^+} & &\\
a_2^{op} &= \sqrt{\frac{g}{2M_2}}\eta_2^{op} & b_2^{op} &= -ie^{ip_1}\sqrt{\frac{g}{2M_2}}\frac{1}{\eta_2^{op}}\left(\frac{x_2^+}{x_2^-}-1\right)\\
c_2^{op} &= -e^{-ip_1}\sqrt{\frac{g}{2M_2}}\frac{\eta_2^{op}}{x_2^+} & d_2^{op} &= i\sqrt{\frac{g}{2M_2}}\frac{x_2^+}{\eta_2^{op}}\left(\frac{x_2^-}{x_2^+}-1\right)\\
\eta_2^{op} &= e^{i\frac{p_2}{4}}e^{i\frac{p_1}{2}}\sqrt{ix_2^- - ix_2^+} & &
\end{aligned}$$

According to the logic of [9], the non-trivial braiding factors present in eq. (2.7) are all hidden in the parameters of the four representations involved.

Using the above described differential representation, we have verified that both $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$ are invariant under Yangian symmetry by explicitly showing that it cocommutes with the above specified coproduct for Yangian generators $\hat{\mathbb{J}}^A$:

$$\mathbb{S}\ \Delta(\hat{\mathbb{J}}^A) = \Delta^{op}(\hat{\mathbb{J}}^A)\ \mathbb{S}. \tag{3.21}$$

Since we work in the evalutation representation, by (2.5), this indeed suffices. We omit the details of the computation since they are not very illuminating.

Most importantly, Yangian symmetry fixes $\mathbb{S}^{BB}$ uniquely up to phase factor without usage of the Yang-Baxter equation. In [44] it was found that by requiring $\mathbb{S}^{BB}$ to be invariant under $\mathfrak{h}$ fixes it up to two coefficients (one being the overall phase which we omit here):

$$\mathbb{S}^{BB} = \mathbb{S}^{BB}_f + q\,\mathbb{S}^{BB}_s. \tag{3.22}$$

The coefficient was then determined by demanding $\mathbb{S}^{BB}$ to satisfy the Yang-Baxter equation [44]. Here, by insisting that the S-matrix (3.22) respects Yangian symmetry we found that this fixes q uniquely and that its value coincides with the one obtained in [44]. This feature of the higher (Yangian) symmetries of the S-matrices as being a substitute for the Yang-Baxter equation is not unexpected, as was explained in [9]. Our computation confirms this point and simultaneously provides an independent check of the results by [44].

4. The near plane-wave limit and the classical r-matrix

We will now concentrate on the plane-wave limit of the bound state S-matrices. In this limit it should agree with the universal classical r-matrix.

4.1 The universal classical r-matrix

In [39] a proposal for the classical r-matrix was made in terms of algebra generators in the evaluation representation which in the classical limit coincides with the S-matrix found in [5, 9].

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The r-matrix is completely given in terms of algebra generators and evaluation parameters u_1, u_2. Consider the following two-site operator

$$\mathcal{T}_{12} = 2\left(\mathbb{R}_\beta{}^\alpha \otimes \mathbb{R}_\alpha{}^\beta - \mathbb{L}_b{}^a \otimes \mathbb{L}_a{}^b + \mathbb{G}_a{}^\alpha \otimes \mathbb{Q}_\alpha{}^a - \mathbb{Q}_\alpha{}^a \otimes \mathbb{G}_a{}^\alpha\right). \tag{4.1}$$

Next we introduce an operator $\mathbb{B}$, which is subject to the following relations (in the classical limit)

$$\begin{aligned} [\mathbb{B}_m, (\mathbb{Q}_n)^a{}_\beta] &= -(\mathbb{Q}_{m+n})^a{}_\beta + 2\epsilon_{\beta\gamma}\epsilon^{ad}(\mathbb{G}_{m+n-1})^\gamma{}_d \\ [\mathbb{B}_m, (\mathbb{G}_n)^\alpha{}_b] &= (\mathbb{G}_{m+n})^\alpha{}_b - 2\epsilon^{\alpha\gamma}\epsilon_{bd}(\mathbb{Q}_{m+n-1})^d{}_\gamma \\ [\mathbb{B}_m, (\mathbb{L}_n)^a{}_b] &= \ [\mathbb{B}_m, (\mathbb{R}_n)^\alpha{}_\beta] = \ [\mathbb{B}_m, (\mathbb{H}_n)] = 0. \end{aligned} \tag{4.2}$$

The action of $\mathbb{B}$ on the fundamental representation should be equal to the action of $\mathcal{T}\mathbb{H}^{-1}$. Finally we would like to note that in the classical limit $u\mathbb{C} = u\mathbb{C}^\dagger = \mathbb{H}$, just as in [39].

In terms of the operator $\mathbb{B}$, the proposed classical r-matrix is [39]

$$r_{12} = \frac{\mathcal{T}_{12} - \mathbb{B}\otimes\mathbb{H} - \mathbb{H}\otimes\mathbb{B}}{i(u_1 - u_2)} - \frac{\mathbb{B}\otimes\mathbb{H}}{iu_2} + \frac{\mathbb{H}\otimes\mathbb{B}}{iu_1} + \frac{i}{2}(u_2^{-1} - u_1^{-1})\mathbb{H}\otimes\mathbb{H}. \tag{4.3}$$

We already know the realization of all the algebra generators on the bound state representations, except for $\mathbb{B}$. The operator $\mathbb{B}$ is characterized through its commutation relations with the generators of $\mathfrak{h}$. It also coincides with $\mathcal{T}\mathbb{H}^{-1}$ on the fundamental representation $M = 1$. An apparent guess would be to identify $\mathbb{B}$ with $\mathcal{T}\mathbb{H}^{-1}$ on the higher representations as well. One should note, however, that this choice is not unique. One can add to $\mathcal{T}\mathbb{H}^{-1}$ the Casimir operator $\mathfrak{C}$ without spoiling any of the commutation relations (4.2). On the fundamental representation the Casimir vanishes and $\mathbb{B}$ coincides with $\mathcal{T}\mathbb{H}^{-1}$. It appears that the correct identification corresponds to taking $\mathbb{B} = \Sigma = \mathcal{T}\mathbb{H}^{-1} + \mathfrak{C}\mathbb{H}^{-1}$. As we will see, this will lead to a complete agreement with the bound state S-matrices in the near plane-wave limit.

Thus, from now on, we will be working with the following r-matrix

$$r_{12} = \frac{\mathcal{T}_{12} - \Sigma\otimes\mathbb{H} - \mathbb{H}\otimes\Sigma}{i(u_1 - u_2)} - \frac{\Sigma\otimes\mathbb{H}}{iu_2} + \frac{\mathbb{H}\otimes\Sigma}{iu_1} + \frac{i}{2}(u_2^{-1} - u_1^{-1})\mathbb{H}\otimes\mathbb{H}. \tag{4.4}$$

The last term is proportional to the identity operator and is related to the phase factor of the S-matrix. It was shown in [39] that r satisfies a number of properties expected from a classical r-matrix like the classical Yang-Baxter equation.

Via (3.2) it is straightforward to put r into differential operator form since it is completely defined in terms of the algebra generators and central elements. Upon taking the near plane-wave limit, discussed below we can then compare this operator to the S-matrix understood as a differential operator.

Let us give the explicit form of r in terms of differential operators and discuss some of its properties. We will consider operators acting on $\Phi_K(w,\theta)\Phi_M(u,\vartheta)$. The operator $\mathcal{T}_{12}$ is simple since it is composed of two operators acting in different spaces. Writing it out is straightforward:

$$
\begin{aligned}
\mathcal{T}_{12} = {} & (-2w_b u_a + w_a u_b)\frac{\partial^2}{\partial w_a \partial u_b} + (2\theta_\beta \vartheta_\alpha - \theta_\alpha \vartheta_\beta)\frac{\partial^2}{\partial \theta_\alpha \partial \vartheta_\beta} \\
& +2(\mathfrak{a}_1\mathfrak{d}_2 - \mathfrak{b}_2\mathfrak{c}_1)u_a\theta_\alpha \frac{\partial^2}{\partial w_a \partial \vartheta_\alpha} + 2(\mathfrak{a}_2\mathfrak{d}_1 - \mathfrak{b}_1\mathfrak{c}_2)w_a\vartheta_\alpha \frac{\partial^2}{\partial u_a \partial \theta_\alpha} \\
& +2(\mathfrak{a}_2\mathfrak{c}_1 - \mathfrak{b}_1\mathfrak{d}_2)\theta_\alpha\vartheta_\beta\epsilon_{ab}\epsilon^{\alpha\beta}\frac{\partial^2}{\partial w_a \partial u_b} + 2(\mathfrak{a}_1\mathfrak{c}_2 - \mathfrak{b}_2\mathfrak{d}_1)w_a u_b \epsilon^{ab}\epsilon_{\alpha\beta}\frac{\partial^2}{\partial \theta_\alpha \partial \vartheta_\beta}. \qquad (4.5)
\end{aligned}
$$

The coefficients $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ are the semi-classical limits of a, b, c, d respectively. Note that the information about the representation is completely encoded in the coefficients a_i, b_i, c_i, d_i as well as in the action of the differential operators on the "short" superfields.

Thus, the explicit form of r depends quite a lot on the choice of the bound state representations. On the other hand, the bound state S-matrices are also quite different from each other and hence the comparison between the two in the classical limit will indeed be a non-trivial check of universality of the proposal.

4.2 The near plane-wave limit

To compare the proposed classical r-matrix to the bound state S-matrices, one first has to define an appropriate limit in which the two can be compared. This limit is called the near plane-wave limit. The observations and analysis done here are similar to those preformed in [50]. Let us first discuss a suitable parameterization of $x^{\pm}_{;M}$ for a M-particle bound state that allows taking the near plane-wave limit. We identify $\hbar = g^{-1}$ and take [51]:

$$
x^{\pm}_{i;M} = x_i \left(\sqrt{1 - \frac{(M/g)^2}{(x_i - \frac{1}{x_i})^2}} \pm \frac{iM/g}{x_i - \frac{1}{x_i}} \right). \qquad (4.6)
$$

By identifying $\hbar = g^{-1}$, it is obvious from (2.6) that to find the classical r-matrix we should expand around $g = \infty$ and work to order g^{-1}.

In this parameterization, most of the parameters simplify greatly. For example, the central charge H is given by

$$
H = M\frac{x^2+1}{x^2-1}. \qquad (4.7)
$$

Crossing symmetry also becomes transparent, since sending $x^{\pm}_i \to \frac{1}{x^{\pm}_i}$ reduces to

$$
x_i \to \frac{1}{x_i}. \qquad (4.8)
$$

This simplifies checking crossing symmetry for the phases encountered later on.

4.3 The dressing phase

The phase factors obtained from fusion and crossing symmetry are given by (3.17). Let us spell them out in the near plane-wave limit since they will come into play when taking

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the semi-classical limit. Consider two bound states of length M_i, M_j, described in the near plane-wave limit by parameters x_i, x_j respectively.

First of all, the functions $G(n)$ and the factors proportional to the momenta are easily expanded around $g \to \infty$ by using (4.6):

$$\begin{aligned} G(n) &= 1 + \frac{2ing^{-1}}{x_1 + \frac{1}{x_1} - x_2 - \frac{1}{x_2}} + \mathcal{O}(g^{-2}) \\ \frac{x_j^+}{x_j^-} &= 1 + 2ig^{-1}M_j \frac{x_j}{x_j^2 - 1} + \mathcal{O}\left(g^{-2}\right). \end{aligned} \tag{4.9}$$

To examine the dressing phase, we first introduce the conserved charges

$$\begin{aligned} q_n(x_i) &= \frac{i}{n-1}\left(\frac{1}{(x_i^+)^{n-1}} - \frac{1}{(x_i^-)^{n-1}}\right) \\ &= 2g^{-1}M_i \frac{x_i^{2-n}}{x_i^2 - 1} + \mathcal{O}(g^{-2}). \end{aligned} \tag{4.10}$$

The dressing phase is related to the conserved charges as follows

$$\sigma(x_i, x_j) = e^{\frac{i}{2}\theta(x_i,x_j)}, \tag{4.11}$$

where

$$\theta_{12} = g\sum_{r=2}^{\infty}\sum_{n=0}^{\infty} c_{r,r+1+2n}\left(q_r\left(x_1\right)q_{r+1+2n}\left(x_2\right) - q_r\left(x_2\right)q_{r+1+2n}\left(x_1\right)\right), \tag{4.12}$$

with [51]

$$c_{r,s} = \delta_{r+1,s} - g^{-1}\frac{4}{\pi}\frac{(r-1)(s-1)}{(r+s-2)(s-r)} + \mathcal{O}(g^{-2}). \tag{4.13}$$

Since $q_n \sim g^{-1}$, we see that if we work to order g^{-1}, it suffices to take $c_{r,s} = \delta_{r+1,s}$. Hence, the dressing phase reduces to

$$\begin{aligned} \theta_{12} &= g\sum_{r=2}^{\infty}\sum_{n=0}^{\infty}\delta_{n,0}\left(q_r\left(x_1\right)q_{r+1+2n}\left(x_2\right) - q_r\left(x_2\right)q_{r+1+2n}\left(x_1\right)\right) + \mathcal{O}(g^{-2}) \\ &= g\sum_{r=2}^{\infty}\left(q_r\left(x_1\right)q_{r+1}\left(x_2\right) - q_r\left(x_2\right)q_{r+1}\left(x_1\right)\right) + \mathcal{O}(g^{-2}) \\ &= 4M_iM_jg^{-1}\frac{x_i^2x_j^2(x_i - x_j)}{(x_i^2-1)(x_j^2-1)}\sum_{r=2}^{\infty}\left(\frac{1}{x_ix_j}\right)^{r+1} + \mathcal{O}(g^{-2}) \\ &= 4M_iM_jg^{-1}\frac{(x_i - x_j)}{(x_i^2-1)(x_ix_j-1)(x_j^2-1)} + \mathcal{O}(g^{-2}). \end{aligned} \tag{4.14}$$

From this expression it is easy to see that, at least to first order, the dressing phases of bound states indeed behave as stated in [44].

For example, consider a two particle bound state, described by momenta p_1, p_2 related by $x_1^- = x_2^+$ and a fundamental excitation with momentum q. From fusion one obtains that the total phase is given by $\theta_{\rm total} = \theta(p_1, q) + \theta(p_2, q)$. However, p_1 and p_2 are not independent, but since $\theta \sim g^{-1}$ we only have to solve the condition $x_1^- = x_2^+$ up to zeroth order, which is easily seen to give $x_1 = x_2 + \mathcal{O}(g^{-1})$. But this means that the phases add and we find $\theta_{\rm total} = 2\theta(p_1, q)$ to first order, which indeed coincides with the found dressing phase.

To conclude, we give the total expression for the complete phase factors (3.17) in the near plane-wave limit:

$$\begin{aligned} S_0^{AA} &= 1 + \frac{i(-1+x_1x_2)(x_1^2+x_2^2)g^{-1}}{(-1+x_1^2)(x_1-x_2)(-1+x_2^2)} + \mathcal{O}(g^{-2}) \\ S_0^{AB} &= 1 + \frac{2i(-1+x_1x_2)(x_1^2+x_2^2)g^{-1}}{(-1+x_1^2)(x_1-x_2)(-1+x_2^2)} + \mathcal{O}(g^{-2}) \\ S_0^{BB} &= 1 + \frac{4i(-1+x_1x_2)(x_1^2+x_2^2)g^{-1}}{(-1+x_1^2)(x_1-x_2)(-1+x_2^2)} + \mathcal{O}(g^{-2}). \end{aligned} \tag{4.15}$$

These phase factors will give a contribution proportional to the identity matrix. We write

$$\mathbb{S} = 1 + g^{-1}\mathbb{S}_{g\to\infty} + \mathcal{O}(g^{-2}). \tag{4.16}$$

4.4 Comparison in the near plane-wave limit

Taking the limit $g \to \infty$ for $\mathbb{S}^{AA}, \mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$, we can compare these matrices with the proposed universal classical r-matrix (4.4). For $\mathbb{S}^{AA}$ this has already been carried out in [39] and complete agreement was found. This is also the case for the discussed bound state S-matrices.

Now we are ready to compare the two operators by considering their action on all basis elements. For all the cases we find a perfect agreement between the limiting values of the S-matrices and the classical r-matrix evaluated in the corresponding bound state representations

$$\mathbb{S}^{AA}_{g\to\infty} = r^{AA}, \qquad \mathbb{S}^{AB}_{g\to\infty} = r^{AB}, \qquad \mathbb{S}^{BB}_{g\to\infty} = r^{BB}. \tag{4.17}$$

Actually, we can do a bit more by comparing r to the proposed phase [44] of the bound state S-matrix $\mathbb{S}^{KM}$ corresponding to the scattering of bound states of length K and M. To this end, we recall that the bound state S-matrices $\mathbb{S}^{KM}$ can be canonically normalized by setting the coefficient a_1, which corresponds to the projector on the irrep with maximal $\mathfrak{su}(2)$ spin, equal to unity:

$$\mathbb{S}^{KM}_{can} w_1^K u_1^M = w_1^K u_1^M. \tag{4.18}$$

For the fully dressed S-matrix we, therefore, obtain

$$\mathbb{S}^{KM} w_1^K u_1^M = S_0^{KM} w_1^K u_1^M, \tag{4.19}$$

where S_0^{KM} is a scalar factor given by [44]:

$$\begin{aligned} S_0^{KM}(x_1,x_2) &= e^{\frac{a}{2}(p_1\epsilon_2 - \epsilon_1 p_2)} \left(\frac{x^-_{1;K}}{x^+_{1;K}}\right)^{\frac{M}{2}} \left(\frac{x^+_{2;M}}{x^-_{2;M}}\right)^{\frac{K}{2}} \sigma(x_1,x_2)\, \times \\ &\quad \times \sqrt{G(M-K)G(M+K)} \prod_{l=1}^{K-1} G(M-K+2l). \end{aligned} \tag{4.20}$$

In the near plane-wave limit, this becomes:

$$S_0^{KM}(x_1,x_2) = 1 + iKM\frac{(x_1x_2-1)(x_1^2+x_2^2)}{(x_1^2-1)(x_1-x_2)(x_2^2-1)}g^{-1} \\ -a\frac{KM\,(x_1-x_2)\,(x_1x_2-1)}{\left(x_1^2-1\right)\left(x_2^2-1\right)}g^{-1} + \mathcal{O}(g^{-2}). \tag{4.21}$$

The piece proportional to a can be realized as an operator

$$-a(u_1^{-1}-u_2^{-1})\mathbb{H}\otimes\mathbb{H}\,. \tag{4.22}$$

On the other hand, assuming that the classical r-matrix is universal, we can easily compute its action on the state $w_1^K u_1^M$. For $a=0$ we find

$$(1+g^{-1}r)w_1^K u_1^M = S_0^{KM}(x_1,x_2)w_1^K u_1^M. \tag{4.23}$$

This means that the phase factor (4.20) derived in [44] is indeed compatible with r. With our choice of $\mathbb{B}=\Sigma$, the proposed r-matrix [39] exhibits perfect "universality" in the sense that it is capable of reproducing the semiclassical limit of the quantum bound state S-matrices $\mathbb{S}^{AA}, \mathbb{S}^{AB}, \mathbb{S}^{BB}$. In particular, it correctly reproduces the semi-classical limit of the quantum phase S_0^{KM} obtained from the fusion procedure.

A last observation is that the form of the r-matrix is quite simple and contains at most three derivatives, whereas an arbitrary S-matrix $\mathbb{S}^{MN}$ of M,N bound states would be build up out of more complicated expressions containing higher order differential operators. This leads to the idea that one could use the proposed r-matrix to identify the non-trivial components of the matrices $\mathbb{S}^{MN}$ and hopefully gain new insights in their structure.

5. Conclusions

We have shown that the recently found bound state S-matrices [44], $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$ are invariant under Yangian symmetry. In particular, Yangian invariance fixes $\mathbb{S}^{BB}$ completely without appealing to the Yang-Baxter equation.

We have also compared the bound state S-matrices in the near plane-wave limit to the proposed universal classical r-matrix of [39]. We found perfect agreement. It would be also interesting to carry out an analogous investigation for the r-matrix proposed in [38].

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